GROUP THEORY 2024 - 25, SOLUTION SHEET 13

Exercise 1. To do yourself. Ask the assistant if something is unclear.

Exercise 2. We will show that a functor $F : BH \to BG$ corresponds precisely to a group homomorphism $H \to G$. We can prove this formally by constructing a bijection

$$\mathrm{Fun}(BH, BG) \xrightarrow{\cong} \mathrm{Hom}_{Gr}(H, G) \tag{1}$$

Since the set of object of BG is a singleton, i.e. $Ob(BG) = \{\bullet_G\}$, and similarly for BH, the data of a functor $F : BH \to BG$ is precisely:

1. a function $F_{Ob} : \{\bullet_H\} \to \{\bullet_G\}$; which necessarily sends $\bullet_H \mapsto \bullet_G$
2. a function $F_{\bullet_H,\bullet_H} : Mor_{BH}(\bullet_H, \bullet_H) \to Mor_{BG}(\bullet_G, \bullet_G)$

such that the following two conditions are satisfied:

a) $F_{\bullet_H,\bullet_H}(Id_{\bullet_H}) = Id_{\bullet_G}$;
b) $F_{\bullet_H,\bullet_H}(g \circ f) = F_{\bullet_H,\bullet_H}(g) \circ F_{\bullet_H,\bullet_H}(f)$ for all $f, g \in Mor_{BH}(\bullet_H, \bullet_H)$.

Now by definition $Mor_{BH}(\bullet_H, \bullet_H) = H$ and $Mor_{BH}(\bullet_G, \bullet_G) = G$ we actually have that the function $F_{\bullet_H,\bullet_H}$ is a function

$$F_{\bullet_H,\bullet_H} : H \to G$$

We claim that this function is actually a group homomorphism. Since the identity in BH of the object $\bullet_H$ is given by $Id_{\bullet_H} = e_H \in Mor_{BH}(\bullet_H, \bullet_H)$, and similarly in BG, condition $a)$ above tells us that F preserves identities

$$F_{\bullet_H,\bullet_H}(e_H) = e_G.$$

Moreover, since composition in BH is given by multiplication in H, and similarly in BG, condition $b)$ implies that $F_{\bullet_H,\bullet_H}$ preserves composition: for all $h, h' \in H$

$$F_{\bullet_H,\bullet_H}(h \cdot h') = F_{\bullet_H,\bullet_H}(h \circ h') = F_{\bullet_H,\bullet_H}(h) \circ F_{\bullet_H,\bullet_H}(h') = F_{\bullet_H,\bullet_H}(h) \cdot F_{\bullet_H,\bullet_H}(h')$$

Together this prove that $F_{\bullet_H,\bullet_H}$ is a group homomorphism. The bijection (1) is defined by

$$(F : BH \to BG) \mapsto (F_{\bullet_H,\bullet_H} : H \to G)$$

Its inverse sends a group homomorphism $f : H \to G$ to the functor $F : BH \to BG$ which on objects is $F(\bullet_H) = \bullet_G$ and on morphisms is $F_{\bullet_H,\bullet_H}(h) = f(h)$. Again the fact that f preserves the neutral elements implies that F satisfies condition $a)$, while the fact that f preserves multiplication implies that F preserves composition.

Exercise 3. We show that such a functor does not exist. Assuming on the contrary; Observe that we have group homomorphisms

$$f : \mathbb{Z}/2\mathbb{Z} \to S_3,\ \overline{1} \mapsto (12)$$

and

$$g : S_3 \to \mathbb{Z}/2\mathbb{Z},\ \sigma \mapsto sgn(\sigma).$$

Note that g is the sign homomorphism which maps an even permutation to $\overline{0}$ and an odd permutations to $\overline{1}$.

Furthermore, one sees that $g \circ f = Id_{\mathbb{Z}/2\mathbb{Z}}$. If Z is a functor with the property that at the level of objects Z assigns a group to it's centre, then on one hand we have that

$$Z(g \circ f) = Z(Id_{\mathbb{Z}/2\mathbb{Z}}) = Id_{\mathbb{Z}/2\mathbb{Z}}.$$

On the other hand since we would have that $Z(S_3) = \{e\}$ we have that $Z(g) \circ Z(f)$ is the composition

$$\mathbb{Z}/2\mathbb{Z} \xrightarrow{Z(f)} \{e\} \xrightarrow{Z(g)} \mathbb{Z}/2\mathbb{Z}$$

which is the trivial homomorphism. Hence $Z(g \circ f)$ is not equal to $Z(g) \circ Z(f)$ and hence Z is not a functor.

Exercise 4. (1) $\Rightarrow$ Assume that f is an epimorphism. Suppose, for contradiction, that f is not surjective. Then there exists an element $y_0 \in Y$ such that $y_0 \notin \operatorname{Im}(f)$. Let

$$Z := \{0, 1\}$$

and define functions $g_1, g_2 : Y \to Z$ by

$$g_1(y) = 0 \quad \text{for all } y \in Y,$$

$$g_2(y) = \begin{cases} 1 & \text{if } y = y_0, \\ 0 & \text{otherwise.} \end{cases}$$

Then $g_1 \neq g_2$, since $g_1(y_0) \neq g_2(y_0)$. However, for every $x \in X$ we have $f(x) \neq y_0$, hence

$$g_1(f(x)) = g_2(f(x)) = 0.$$

Thus $g_1 \circ f = g_2 \circ f$, contradicting the assumption that f is an epimorphism. Therefore f must be surjective.

$\Leftarrow$ Conversely, assume that f is surjective. Let $g_1, g_2 : Y \to Z$ be functions such that $g_1 \circ f = g_2 \circ f$. For any $y \in Y$, surjectivity of f implies the existence of $x \in X$ with $f(x) = y$. Then

$$g_1(y) = g_1(f(x)) = g_2(f(x)) = g_2(y).$$

Hence $g_1 = g_2$, and f is an epimorphism.

(2) $\Rightarrow$ Assume that f is a monomorphism. Suppose that f is not injective. Then there exist distinct elements $x_1, x_2 \in X$ such that

$$f(x_1) = f(x_2).$$

Let $Z := \{*\}$ be a singleton set, and define functions $h_1, h_2 : Z \to X$ by

$$h_1(*) = x_1, \qquad h_2(*) = x_2.$$

Then $h_1 \neq h_2$, but

$$f \circ h_1 = f \circ h_2,$$

contradicting the assumption that f is a monomorphism. Therefore f must be injective.

$\Leftarrow$ Conversely, assume that f is injective. Let $h_1, h_2 : Z \to X$ be functions such that $f \circ h_1 = f \circ h_2$. For any $z \in Z$, we have

$$f(h_1(z)) = f(h_2(z)).$$

Since f is injective, this implies $h_1(z) = h_2(z)$ for all $z \in Z$, and hence $h_1 = h_2$. Thus f is a monomorphism.

Exercise 5. By definition, a pushout of (f, f') consists of an object P together with morphisms

$$i : Y \to P, \qquad i' : Y' \to P$$

such that the square

$$\begin{tikzcd}
X \arrow[r, "f'"] \arrow[d, "f"'] & Y' \arrow[d, "i'"] \\
Y \arrow[r, "i"'] & P
\end{tikzcd}$$

commutes, i.e. $i \circ f = i' \circ f'$, and which is universal with respect to this property: for any object Z and morphisms

$$g : Y \to Z, \qquad g' : Y' \to Z$$

satisfying $g \circ f = g' \circ f'$, there exists a unique morphism $u : P \to Z$ such that

$$u \circ i = g, \qquad u \circ i' = g'.$$

Suppose now that (P, i, i') and (P', j, j') are two pushouts of the given diagram. We will show that P and P' are isomorphic.

Since (P', j, j') is a pushout and the morphisms

$$i : Y \to P, \qquad i' : Y' \to P$$

satisfy $i \circ f = i' \circ f'$, the universal property of P' yields a unique morphism

$$\alpha : P' \to P$$

such that

$$\alpha \circ j = i, \qquad \alpha \circ j' = i'.$$

Similarly, since (P, i, i') is a pushout and

$$j : Y \to P', \qquad j' : Y' \to P'$$

satisfy $j \circ f = j' \circ f'$, there exists a unique morphism

$$\beta : P \to P'$$

such that

$$\beta \circ i = j, \qquad \beta \circ i' = j'.$$

We now show that α and β are inverse to each other. Consider the composition $\alpha \circ \beta : P \to P$. We have

$$(\alpha \circ \beta) \circ i = \alpha \circ (\beta \circ i) = \alpha \circ j = i,$$

and similarly

$$(\alpha \circ \beta) \circ i' = \alpha \circ (\beta \circ i') = \alpha \circ j' = i'.$$

By the universal property of the pushout (P, i, i'), there is a unique endomorphism of P with this property. Since the identity morphism id_P also satisfies

$$\mathrm{id}_P \circ i = i, \qquad \mathrm{id}_P \circ i' = i',$$

it follows that $\alpha \circ \beta = \mathrm{id}_P$. An analogous argument shows that $\beta \circ \alpha = \mathrm{id}_{P'}$.

Therefore $\alpha : P' \to P$ is an isomorphism with inverse $\beta : P \to P'$. Hence the pushout, if it exists, is unique up to isomorphism.

Exercise 6. **(a)** The key observation is, given an element $x \in X$, we get always a map $G \to X, g \mapsto g \cdot x$, and this should induce a function $G \to H(Y), g \mapsto g \cdot f(x) = f(g \cdot x)$. Therefore we define

$$\begin{aligned} H(Y) &:= \{\text{Maps } \varphi : G \to Y\}, \\ g \cdot \varphi &:= h \mapsto \varphi(h \cdot g). \end{aligned}$$

It is easy to verify that this builds G-set. Given a map $f : F(X) \to Y$ between sets, we can define

$$\begin{aligned} X &\to H(Y), \\ x &\mapsto (h \mapsto f(h \cdot x)). \end{aligned}$$

We omit the verification that this is a map between G-sets. Conversely, give a map $m : X \to H(Y)$ of G-sets, we can define

$$\begin{aligned} F(X) &\to Y, \\ x &\mapsto (m(x))(e). \end{aligned}$$

We omit again the verification that the two constructions are mutually inverse.

(a)

We claim that the right adjoint is $Y \mapsto Y^G$, the fixed point set of Y. Indeed, assume that there is a map $f : \mathrm{Triv}(X) \mapsto Y$, then we have $f(\mathrm{Triv}(X)) \subset Y^G$ since $g \cdot f(x) = f(g \cdot x) = f(x)$ for all $x \in X$. Conversely, a map of sets $X \to Y^G$ along with the embedding $Y^G \to Y$ gives a map $X \to Y$ and one can check this gives a map of G-sets $\mathrm{Triv}(X) \to Y$. We omit the verification of mutual inverses.